Int. J. Solids Structures Vol. 23, No. 1, pp. 111–131, 1987
Printed in Great Britain

0020-7683/87 $3.00 + .00
Pergamon Journals Ltd.

A HIGH-ORDER LAMINATED PLATE THEORY WITH IMPROVED IN-PLANE RESPONSES†

A. TOLEDANO‡ and H. MURAKAMI§
Department of Applied Mechanics and Engineering Sciences,
University of California at San Diego, La Jolla, CA 92093, U.S.A.

(*Received* 14 *June* 1985; *in revised form* 8 *March* 1986)

Abstract—In order to improve the accuracy of the in-plane responses of the shear deformable laminated composite plate theories, a new high-order laminated plate theory was developed based upon Reissner's new mixed variational principle [*Int. J. Num. Meth. Eng.* **20**, 1366 (1984)]. To this end, a zig-zag shaped C^0 function and Legendre polynomials were introduced into the approximate in-plane displacement distributions across the plate thickness. The accuracy of the present theory was examined by applying it to the cylindrical bending problem of laminated plates which had been solved exactly by Pagano [*J. Comp. Mat.* **3**, 398 (1969)]. A comparison with the exact solutions obtained for several symmetric and asymmetric cross-ply laminates indicates that the present theory accurately estimates in-plane responses, even for small span-to-thickness ratios

1. INTRODUCTION

The increasing use of composite materials as thick laminates, in aerospace engineering and in automotive engineering, has clearly demonstrated the need for the development of new theories to efficiently and accurately predict the behavior of such structural components. The intrinsic heterogeneity and anisotropy of these composite structures as evidenced in the stacking of several fibrous layers and in the high discontinuity in material properties across the interfaces, make the classical theories of plates and shells inadequate.

The inspiration and guidelines for the subsequent attempts have stemmed from Pagano's works[1–3] where the exact elasticity solutions for the problems of cylindrical bending and simply supported rectangular plates were given. Pagano showed the importance of incorporating the effect of transverse shear deformations in order to accurately estimate the plate lateral deflection and the need to improve upon the thickness variation of the in-plane displacements, which are assumed to be C^1 linear functions in both classical plate theory (CPT) and Reissner–Mindlin plate theory (FSD).

The first attempt to develop a general linear laminated plate theory is credited to Yang, Norris and Stavsky[4]. Their theory is an extension of the Reissner–Mindlin homogeneous plate theory as applied to an arbitrary number of bonded anisotropic layers. Whitney and Pagano[5] extended Yang, Norris and Stavsky's work. An important conclusion drawn from their analysis, which was also emphasized later by Whitney[6], is that the inaccuracies of the classical plate theory at low span-to-thickness ratios for determining in-plane stresses are not alleviated by the introduction of shear deformations. Whitney[6] obtained in-plane displacements by integrating the transverse shear strains deduced in [5]. This resulted in a higher order approximation which accurately predicted in-plane strains, but the resulting modified stresses did not necessarily satisfy the original plate equilibrium equations.

Since then, other high-order laminated plate theories have been proposed that account for transverse shear strains. Of these, the Lo, Christensen and Wu[7] and the Reddy[8] high-order models have served as the foundation for the present theory. In their paper[7], Lo, Christensen and Wu used appropriate higher order terms in the power series expansions of the assumed displacement field which was proposed by Hildebrand, Reissner and

† Research was supported by the Office of Naval Research under Contract N00014-84-K-0468 to the University of California at San Diego.
‡ Research Assistant.
§ Associate Professor of Applied Mechanics.

Thomas[9]. On the other hand, Reddy[8] imposed the condition of vanishing transverse shear strains on the top and bottom surfaces of the plate. However, this theory does not satisfy the continuity condition of transverse shear stresses at the interfaces.

The objective of the present paper is to improve the approximation of in-plane variables in laminated plate theories. In-plane displacements and bending and stretching stresses are considered primary quantities in any approximate laminated plate analysis; transverse stresses are only of secondary importance since they are an order of magnitude smaller than the primary bending and stretching stresses. By using a new mixed variational principle proposed by Reissner[10], the present theory is a high-order model which improves upon existing theories by including in the assumed in-plane displacement variations across the plate thickness: (1) a zig-zag shaped C^0 function as detailed by Murakami[12]; and (2) Legendre polynomials. The advantage of using Reissner's new mixed variational principle is that it automatically yields the appropriate shear correction factors for the transverse shear constitutive equations. Another attractive feature of the proposed theory is that the number of equations to be solved is not increased as the number of layers becomes larger and larger. A comparison of the proposed theory with Pagano's exact elasticity solution for symmetric and asymmetric laminated plates in cylindrical bending, shows that in-plane displacements and stresses are accurately predicted by the inclusion of the zig-zag shaped function and the Legendre polynomials.

2. FORMULATION

Consider an N-layer laminated composite plate, shown in Fig. 1, with principal axes coinciding with a Cartesian coordinate system (x_1, x_2, x_3), such that the x_3-axis is perpendicular to the plane defined by x_1 and x_2. The following notation: $(\ )^k$, $k = 1, 2, \ldots, N$ will designate quantities associated with the kth layer. The thickness of each layer is $n^{(k)}h$, where h is the total thickness of the plate. The volume fractions $n^{(k)}$ satisfy the relation

$$\sum_{k=1}^{N} n^{(k)} = 1. \tag{1}$$

Unless otherwise specified, the usual Cartesian indicial notation is employed where Latin

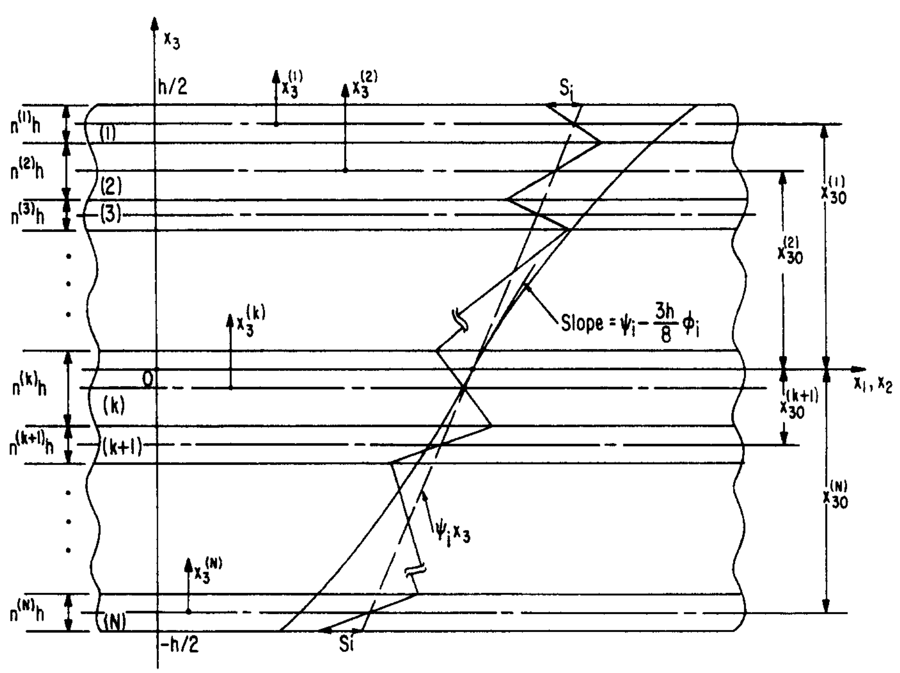

Fig. 1. Plate geometry, coordinate system and trial in-plane displacements.

and Greek indices range from 1 to 3 and 1 to 2, respectively. Repeated indices imply the summation convention and $(\ )_{,i}$ is used to denote partial differentiation with respect to x_i.

With the help of the foregoing notation, the governing equations for the displacement vector $u_i^{(k)}$ and stress tensor $\sigma_{ij}^{(k)}$ associated with the kth layer are:

(a) Equilibrium equations

$$\sigma_{ji,j}^{(k)}+f_i^{(k)}=0;\quad \sigma_{ij}^{(k)}=\sigma_{ji}^{(k)} \tag{2}$$

where f_i are the body forces.

(b) Constitutive equations for orthotropic layers

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{bmatrix}^{(k)}=\begin{bmatrix}\tilde{C}_{11} & \tilde{C}_{12} & 0\\ \tilde{C}_{12} & \tilde{C}_{22} & 0\\ 0 & 0 & \tilde{C}_{66}\end{bmatrix}^{(k)}\begin{bmatrix}e_{11}\\ e_{22}\\ 2e_{12}\end{bmatrix}^{(k)}+\begin{bmatrix}C_{13}/C_{33}\\ C_{23}/C_{33}\\ 0\end{bmatrix}^{(k)}\sigma_{33}^{(k)} \tag{3a}$$

$$\begin{bmatrix}e_{33}\\ 2e_{23}\\ 2e_{31}\end{bmatrix}^{(k)}=-\begin{bmatrix}C_{13}/C_{33} & C_{23}/C_{33} & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}^{(k)}\begin{bmatrix}e_{11}\\ e_{22}\\ 2e_{12}\end{bmatrix}^{(k)}+\begin{bmatrix}1/C_{33} & 0 & 0\\ 0 & 1/C_{44} & 0\\ 0 & 0 & 1/C_{55}\end{bmatrix}^{(k)}\begin{bmatrix}\sigma_{33}\\ \sigma_{23}\\ \sigma_{31}\end{bmatrix}^{(k)} \tag{3b}$$

where C_{ij} are the elastic constants and $\tilde{C}_{ij}$ $(i,j=1,2,6)$ represent the reduced stiffnesses introduced by Whitney and Pagano[5].

(c) Strain–displacement relations

$$e_{ij}^{(k)}=\tfrac{1}{2}(u_{i,j}^{(k)}+u_{j,i}^{(k)}). \tag{4}$$

(d) Interface continuity conditions

$$u_i^{(k)}=u_i^{(k+1)},\quad \sigma_{3i}^{(k)}=\sigma_{3i}^{(k+1)};\quad k=1,2,\ldots,N-1. \tag{5}$$

(e) Upper and lower surface stress conditions

$$\sigma_{3i}^{(1)}=T_i^{+}\quad \text{on}\quad x_3=\frac{h}{2} \tag{6a}$$

$$\sigma_{3i}^{(N)}=T_i^{-}\quad \text{on}\quad x_3=-\frac{h}{2}. \tag{6b}$$

The objective in developing a new laminated plate theory is twofold: first, to improve the assumed variation of in-plane displacements through the thickness of the plate and second, to include the effect of transverse shear deformation. In order to carry out this task, Reissner's new mixed variational principle[10] was applied to the N-layer composite plate whose middle surface occupies a domain D in the x_1, x_2-plane:

$$\begin{aligned}&\iint_D\left[\sum_k\int_{A^{(k)}}\{\delta e_{ij}^{(k)}\sigma_{ij}^{(k)}+[u_{\alpha,3}^{(k)}+u_{3,\alpha}^{(k)}-2e_{3\alpha}^{(k)}(\cdots)]\,\delta\tau_{3\alpha}^{(k)}\right.\\&\left.\quad+[u_{3,3}^{(k)}-e_{33}^{(k)}(\cdots)]\,\delta\tau_{33}^{(k)}\}\,\mathrm{d}x_3\right]\mathrm{d}x_1\,\mathrm{d}x_2\\&=\iint_D\left[\sum_k\int_{A^{(k)}}\delta u_i^{(k)}f_i^{(k)}\,\mathrm{d}x_3\right]\mathrm{d}x_1\,\mathrm{d}x_2+\int_{\partial D_T}\left[\sum_k\int_{A^{(k)}}\delta u_i^{(k)}\check{T}_i^{(k)}\,\mathrm{d}x_3\right]\mathrm{d}s\\&\quad+\iint_D\left[\delta u_i^{(1)}\left(x_1,x_2,\frac{h}{2}\right)T_i^{+}-\delta u_i^{(N)}\left(x_1,x_2,-\frac{h}{2}\right)T_i^{-}\right]\mathrm{d}x_1\,\mathrm{d}x_2\end{aligned} \tag{7}$$

where ∂D_T denotes the boundary of D with outward normal v_α on which tractions $\bar{T}_i$ are specified and $A^{(k)}$ represents the x_3-domain occupied by the kth layer. Also $\tau_{3i}^{(k)}$ denote the approximate transverse stresses and $e_{3i}^{(k)}(\cdots)$ implies the appropriate right-hand side of (3b). The significance of eqn (7) lies in the fact that it is a mixed variational principle for displacements and *transverse* stresses only. This means that eqns (2a) and (3b) are obtained as the Euler–Lagrange equations of (7), while eqn (3a) are considered to be the definitions of $\sigma_{\alpha\beta}^{(k)}$ and $\delta e_{ij}^{(k)}$ is determined by taking the variation of eqn (4). This is the reason why the constitutive equations of three-dimensional elasticity are written in the form (3a, b), as was shown by Reissner[10, 11]. It now becomes clear that, for laminated plate problems, it allows to make approximative assumptions regarding transverse stresses which are continuous across the plate thickness and an order of magnitude smaller than the primary bending and stretching stresses.

3. TRIAL DISPLACEMENT FIELD, TRANSVERSE AND NORMAL STRESSES

The high-order laminated plate theory which takes into account the effect of transverse shear strains, is obtained by including the Legendre polynomials of order $n = 1, 2, 3$ with respect to the x_3-coordinate to a zig-zag in-plane displacement variation of amplitude $S_i(x_1, x_2)$ across the plate thickness.

The appropriate trial functions used in connection with Reissner's mixed variational principle eqn (7) are taken to be:

(a) Trial displacement field

$$u_i^{(k)}(x_1, x_2, x_3) = U_i(x_1, x_2) + \left(\frac{h}{2}\right)\Psi_i(x_1, x_2)P_1(\zeta) + S_i(x_1, x_2)(-1)^k \frac{2}{n^{(k)}h} x_3^{(k)} + \left(\frac{h}{2}\right)^2 \xi_i(x_1, x_2)P_2(\zeta) + \left(\frac{h}{2}\right)^3 \phi_i(x_1, x_2)P_3(\zeta) \quad (8)$$

where $\zeta \equiv \frac{2x_3}{h}$ and $P_n(\zeta)$ are the Legendre polynomials of order n. It is also understood that $\phi_3 \equiv 0$. $x_3^{(k)}$ is a local x_3-coordinate system with its origin at the center $x_{30}^{(k)}$ of the kth layer, i.e.

$$x_3^{(k)} \equiv x_3 - x_{30}^{(k)}. \quad (9)$$

Equation (8) may be regarded as a superposition of a zig-zag function and the cubic variation as proposed by Lo, Christensen and Wu[7], with the exception that here Legendre polynomials are used instead of simple powers in x_3.

(b) Trial transverse and normal stresses

$$\tau_{3\alpha}^{(k)}(x_1, x_2, x_3) = Q_\alpha^{(k)}(x_1, x_2)F_1(z) + R_\alpha^{(k)}(x_1, x_2)F_2(z) + J_\alpha^{(k)}(x_1, x_2)F_3(z) + [T_\alpha^{(k-1)}(x_1, x_2) + T_\alpha^{(k)}(x_1, x_2)]F_4(z) + [T_\alpha^{(k-1)}(x_1, x_2) - T_\alpha^{(k)}(x_1, x_2)]F_5(z); \quad (10a)$$

$$\tau_{33}^{(k)}(x_1, x_2, x_3) = Q_3^{(k)}(x_1, x_2)F_1(z) + R_3^{(k)}(x_1, x_2)F_6(z) + J_3^{(k)}(x_1, x_2)F_3(z) + I_3^{(k)}(x_1, x_2)F_7(z) + [T_3^{(k-1)}(x_1, x_2) + T_3^{(k)}(x_1, x_2)]F_4(z) + [T_3^{(k-1)}(x_1, x_2) - T_3^{(k)}(x_1, x_2)]F_8(z) \quad (10b)$$

where

$$F_1(z) = \frac{5}{n^{(k)}h}\left(21z^4 - \frac{15}{2}z^2 + \frac{9}{16}\right), \qquad F_2(z) = \frac{-30}{(n^{(k)}h)^2}(4z^3 - z)$$

$$F_3(z) = \frac{-105}{(n^{(k)}h)^3}\left(20z^4 - 6z^2 + \frac{1}{4}\right), \qquad F_4(z) = 35z^4 - \frac{15}{2}z^2 + \frac{3}{16}$$

$$F_5(z) = 10z^3 - \frac{3}{2}z, \qquad F_6(z) = \frac{105}{(n^{(k)}h)^2}\left(36z^5 - 14z^3 + \frac{5}{4}z\right) \tag{11}$$

$$F_7(z) = \frac{-315}{(n^{(k)}h)^4}(112z^5 - 40z^3 + 3z), \qquad F_8(z) = 126z^5 - 35z^3 + \frac{15}{8}z$$

$$\text{and} \qquad z \equiv \frac{x_3^{(k)}}{n^{(k)}h}, \qquad -\frac{1}{2} \leqslant z \leqslant \frac{1}{2}.$$

Also,

$$(Q_i^{(k)}, R_i^{(k)}, J_i^{(k)}) \equiv \int_{A^{(k)}} (1, x_3^{(k)}, x_3^{(k)2})\tau_{3i}^{(k)}\, dx_3 \tag{12a}$$

$$I_3^{(k)} \equiv \int_{A^{(k)}} x_3^{(k)3}\tau_{33}^{(k)}\, dx_3. \tag{12b}$$

In (10) $T_i^{(k-1)}$ and $T_i^{(k)}$ are the values of $\tau_{3i}^{(k)}$ at the top and bottom surfaces of the kth layer, respectively. From (6)

$$T_i^{(0)} = T_i^+ \qquad \text{and} \qquad T_i^{(N)} = T_i^-. \tag{13}$$

The functions $F_i(z)$, $i = 1,\dots,8$ are obtained by first noting that eqn (8) yields cubic variations across the plate thickness of in-plane stresses. From the equilibrium equations (2a) transverse stresses $\tau_{3\alpha}^{(k)}$ and $\tau_{33}^{(k)}$ may, as a result, be represented by polynomials of degree 4 and 5 in z, respectively. Their corresponding coefficients are then computed by using eqns (12a, b). This yields the functions $F_i(z)$.

4. LAMINATED PLATE EQUATIONS

Substituting (8) and (10) into (7), using Gauss' theorem and the orthogonality property of the Legendre polynomials one obtains:

(a) Equilibrium equations:

$$N_{\alpha i,\alpha} + T_i^+ - T_i^- + F_i^N = 0 \tag{14a}$$

$$M_{\alpha i,\alpha} - N_{3i} + \frac{h}{2}(T_i^+ + T_i^-) + F_i^M = 0 \tag{14b}$$

$$Z_{\alpha i,\alpha} - K_{3i} - [T_1^+ - (-1)^N T_i^-] + F_i^Z = 0 \tag{14c}$$

$$L_{\alpha i,\alpha} - 3M_{3i} + \frac{h^2}{4}(T_i^+ - T_i^-) + F_i^L = 0 \tag{14d}$$

$$P_{\beta\alpha,\beta} - \left(5L_{3\alpha} + \frac{h^2}{4}N_{3\alpha}\right) + \frac{h^3}{8}(T_\alpha^+ + T_\alpha^-) + F_\alpha^P = 0 \tag{14e}$$

where

$$\begin{bmatrix} N_{\alpha\beta}, & M_{\alpha\beta}, & Z_{\alpha\beta}, & L_{\alpha\beta}, & P_{\alpha\beta} \\ F_i^N, & F_i^M, & F_i^Z, & F_i^L, & F_i^P \end{bmatrix} \equiv \sum_{k=1}^{N} \int_{A^{(k)}} \left[1, \frac{h}{2} P_1(\zeta), (-1)^k \frac{2x_3^{(k)}}{n^{(k)}h}, \left(\frac{h}{2}\right)^2 P_2(\zeta), \left(\frac{h}{2}\right)^3 P_3(\zeta) \right] \begin{bmatrix} \sigma_{\alpha\beta}^{(k)} \\ f_i^{(k)} \end{bmatrix} \mathrm{d}x_3 \quad (15a, b)$$

$$(N_{3i}, M_{3i}, K_{3i}, Z_{3i}, L_{3i}) \equiv \sum_{k=1}^{N} \int_{A^{(k)}} \left[1, \frac{h}{2} P_1(\zeta), (-1)^k \frac{2}{n^{(k)}h}, (-1)^k \frac{2}{n^{(k)}h} x_3^{(k)}, \left(\frac{h}{2}\right)^2 P_2(\zeta) \right] \tau_{3i}^{(k)} \, \mathrm{d}x_3. \quad (15c)$$

(b) Constitutive equations:

(i) for transverse stresses

$$Q_\alpha^{(k)} - \frac{8J_\alpha^{(k)}}{(n^{(k)}h)^2} + \frac{n^{(k)}h}{30}(T_\alpha^{(k-1)} + T_\alpha^{(k)}) = \frac{2}{5} hn^{(k)} \tilde{C}_\alpha^{(k)} \left[U_{3,\alpha} + \Psi_\alpha + S_\alpha (-1)^k \frac{2}{n^{(k)}h} + hn_o^{(k)}(\Psi_{3,\alpha} + 3\xi_\alpha) + \frac{h^2}{2}\left(3n_o^{(k)2} - \frac{1}{4}\right)\xi_{3,\alpha} + \frac{3h^2}{2}\left(5n_o^{(k)2} - \frac{1}{4}\right)\phi_\alpha \right] \quad (16a)$$

$$\frac{1}{h} R_\alpha^{(k)} - \frac{n^{(k)2}h}{40}(T_\alpha^{(k-1)} - T_\alpha^{(k)}) = \frac{7h^2}{120} n^{(k)3} \tilde{C}_\alpha^{(k)} \left[\Psi_{3,\alpha} + 3\xi_\alpha + S_{3,\alpha}(-1)^k \frac{2}{n^{(k)}h} + 3hn_o^{(k)}(\xi_{3,\alpha} + 5\phi_\alpha) \right] \quad (16b)$$

$$Q_\alpha^{(k)} - \frac{14J_\alpha^{(k)}}{(n^{(k)}h)^2} + \frac{n^{(k)}h}{12}(T_\alpha^{(k-1)} + T_\alpha^{(k)}) = -\frac{3h^3}{40} n^{(k)3} \tilde{C}_\alpha^{(k)} (\xi_{3,\alpha} + 5\phi_\alpha) \quad (16c)$$

$$-\frac{1}{\tilde{C}_\alpha^{(k)}}\left[\frac{1}{12} Q_\alpha^{(k)} - \frac{5J_\alpha^{(k)}}{3(n^{(k)}h)^2} + \frac{3R_\alpha^{(k)}}{7n^{(k)}h}\right] - \frac{1}{\tilde{C}_\alpha^{(k+1)}}\left[\frac{1}{12} Q_\alpha^{(k+1)} - \frac{5J_\alpha^{(k+1)}}{3(n^{(k+1)}h)^2} - \frac{3R_\alpha^{(k+1)}}{7n^{(k+1)}h}\right] = \frac{h}{126}\left[\frac{-n^{(k)}}{\tilde{C}_\alpha^{(k)}} T_\alpha^{(k-1)} + 8\left(\frac{n^{(k)}}{\tilde{C}_\alpha^{(k)}} + \frac{n^{(k+1)}}{\tilde{C}_\alpha^{(k+1)}}\right) T_\alpha^{(k)} - \frac{n^{(k+1)}}{\tilde{C}_\alpha^{(k+1)}} T_\alpha^{(k+1)}\right]; \quad (16d)$$

(ii) for normal stresses

$$Q_3^{(k)} - \frac{8J_3^{(k)}}{(n^{(k)}h)^2} + \frac{n^{(k)}h}{30}(T_3^{(k-1)} + T_3^{(k)}) = \frac{2h}{5} n^{(k)} C_{33}^{(k)} \left[\Psi_3 + S_3(-1)^k \frac{2}{n^{(k)}h} + 3hn_o^{(k)}\xi_3 \right] + \frac{2h}{5} n^{(k)} \left[\tilde{U} + hn_o^{(k)}\tilde{\Psi} + \frac{h^2}{2}\left(3n_o^{(k)2} - \frac{1}{4}\right)\tilde{\xi} + \frac{h^3}{2}\left(5n_o^{(k)3} - \frac{3}{4}n_o^{(k)}\right)\tilde{\phi} \right] \quad (17a)$$

$$\frac{1}{h} R_3^{(k)} - \frac{32I_3^{(k)}}{5n^{(k)2}h^3} + \frac{n^{(k)2}h}{140}(T_3^{(k-1)} - T_3^{(k)}) = \frac{11}{350} h^2 n^{(k)3} C_{33}^{(k)} \xi_3 + \frac{11}{1050} h^2 n^{(k)3} \left[\tilde{\Psi} + (-1)^k \frac{2}{n^{(k)}h} \tilde{S} + 3hn_o^{(k)}\tilde{\xi} + \frac{3h^2}{2}\left(5n_o^{(k)2} - \frac{1}{4}\right)\tilde{\phi} \right] \quad (17b)$$

$$Q_3^{(k)} - \frac{14J_3^{(k)}}{(n^{(k)}h)^2} + \frac{n^{(k)}h}{12}(T_3^{(k-1)} + T_3^{(k)}) = -\frac{3h^3}{40}n^{(k)3}[\tilde{\xi} + 5hn_o^{(k)}\tilde{\phi}] \tag{17c}$$

$$\frac{1}{h}R_3^{(k)} - \frac{15I_3^{(k)}}{2n^{(k)2}h^3} + \frac{n^{(k)2}h}{96}(T_3^{(k-1)} - T_3^{(k)}) = \frac{-11h^4}{2688}n^{(k)5}\tilde{\phi} \tag{17d}$$

$$\frac{-11}{12}\left[\frac{Q_3^{(k)}}{C_{33}^{(k)}} + \frac{Q_3^{(k+1)}}{C_{33}^{(k+1)}}\right] + \frac{15}{2h}\left[\frac{R_3^{(k)}}{n^{(k)}C_{33}^{(k)}} - \frac{R_3^{(k+1)}}{n^{(k+1)}C_{33}^{(k+1)}}\right]$$

$$+\frac{55}{3h^2}\left[\frac{J_3^{(k)}}{n^{(k)2}C_{33}^{(k)}} + \frac{J_3^{(k+1)}}{n^{(k+1)2}C_{33}^{(k+1)}}\right] - \frac{70}{h^3}\left[\frac{I_3^{(k)}}{n^{(k)3}C_{33}^{(k)}} - \frac{I_3^{(k+1)}}{n^{(k+1)3}C_{33}^{(k+1)}}\right]$$

$$= \frac{h}{18}\left[\frac{n^{(k)}}{C_{33}^{(k)}}T_3^{(k-1)} + 10\left(\frac{n^{(k)}}{C_{33}^{(k)}} + \frac{n^{(k+1)}}{C_{33}^{(k+1)}}\right)T_3^{(k)} + \frac{n^{(k+1)}}{C_{33}^{(k+1)}}T_3^{(k+1)}\right] \tag{17e}$$

where in (16a, b, c) and (17a, b, c, d) k ranges from 1 to N while in (16d) and (17e) k ranges from 1 to $(N-1)$. Also, no summation on α is implied in (16) and

$$\tilde{C}_\alpha^{(k)} \equiv \delta_{\alpha 1}C_{55}^{(k)} + \delta_{\alpha 2}C_{44}^{(k)}; \qquad n_o^{(k)} \equiv x_{30}^{(k)}/h \tag{18}$$

$$\begin{bmatrix} \tilde{U} \\ \tilde{\Psi} \\ \tilde{S} \\ \tilde{\xi} \\ \tilde{\phi} \end{bmatrix} = \begin{bmatrix} U_{1,1} & U_{2,2} \\ \Psi_{1,1} & \Psi_{2,2} \\ S_{1,1} & S_{2,2} \\ \xi_{1,1} & \xi_{2,2} \\ \phi_{1,1} & \phi_{2,2} \end{bmatrix} \begin{bmatrix} C_{13} \\ C_{23} \end{bmatrix}^{(k)}. \tag{19}$$

By solving (16) and (17), $Q_i^{(k)}$, $R_i^{(k)}$, $J_i^{(k)}$, $I_3^{(k)}$ and $T_i^{(k)}$ are obtained in terms of U_i, Ψ_i, S_i, ξ_i and ϕ_α and their derivatives. As a result, the quantities N_{3i}, M_{3i}, K_{3i}, Z_{3i}, L_{3i} of eqn (15c) can be determined as functions of these displacement variables. Such expressions will automatically include the appropriate shear correction factors by virtue of the Reissner mixed variational principle.

The equilibrium eqns (14) are supplemented with the following suitable boundary conditions:

$$\text{specify } U_i \quad \text{or} \quad N_{\alpha i}\nu_\alpha, \tag{20a}$$

$$\text{specify } \Psi_i \quad \text{or} \quad M_{\alpha i}\nu_\alpha, \tag{20b}$$

$$\text{specify } S_i \quad \text{or} \quad Z_{\alpha i}\nu_\alpha, \tag{20c}$$

$$\text{specify } \xi_i \quad \text{or} \quad L_{\alpha i}\nu_\alpha, \tag{20d}$$

$$\text{specify } \phi_\alpha \quad \text{or} \quad P_{\beta\alpha}\nu_\beta. \tag{20e}$$

The remaining constitutive equations for $N_{\alpha\beta}$, $M_{\alpha\beta}$, $Z_{\alpha\beta}$, $L_{\alpha\beta}$ and $P_{\alpha\beta}$ are obtained by substituting (3a), (4), (8) and (10b) into (15a) to yield :

$$\begin{bmatrix} \frac{1}{h}\underset{\sim}{N} \\ \frac{1}{h^2}\underset{\sim}{M} \\ \frac{1}{h}\underset{\sim}{Z} \\ \frac{1}{h^3}\underset{\sim}{L} \\ \frac{1}{h^4}\underset{\sim}{P} \end{bmatrix} = \begin{bmatrix} [N_u] & [N_\Psi] & 0 & [N_\xi] & [N_\phi] \\ & [M_\Psi] & [M_s] & [M_\xi] & [M_\phi] \\ & & \frac{1}{3}[N_u] & [Z_\xi] & [Z_\phi] \\ \text{symmetric} & & & [L_\xi] & [L_\phi] \\ & & & & [P_\phi] \end{bmatrix} \begin{bmatrix} \underset{\sim}{U} \\ h\underset{\sim}{\Psi} \\ \underset{\sim}{S} \\ h^2\underset{\sim}{\xi} \\ h^3\underset{\sim}{\phi} \end{bmatrix}$$

$$+\frac{1}{h}\sum_{k=1}^{N}[C]^{(k)} \begin{bmatrix} \underset{\sim}{V}^N \\ \underset{\sim}{V}^M \\ \underset{\sim}{V}^Z \\ \underset{\sim}{V}^L \\ \underset{\sim}{V}^P \end{bmatrix} \begin{bmatrix} Q_3 \\ \frac{1}{h}R_3 \\ \frac{1}{h^2}J_3 \\ \frac{1}{h^3}I_3 \end{bmatrix}^{(k)} \tag{21}$$

where $\underset{\sim}{N} = [N_{11}, N_{22}, N_{12}]^T$, $\underset{\sim}{U} = [U_{1,1}, U_{2,2}, U_{1,2}+U_{2,1}]^T$ with analogous expressions for $\underset{\sim}{M}, \underset{\sim}{\Psi}, \ldots, \underset{\sim}{P}, \underset{\sim}{\phi}$. $[N_u], \ldots, [P_\phi]$ are 3×3 matrices, $[C]^{(k)}$ is a 15×5 matrix and $\underset{\sim}{V}^N, \ldots, \underset{\sim}{V}^P$ are 1×4 vectors, which are given in the Appendix.

5. CYLINDRICAL BENDING OF LAMINATED PLATES

In order to test the accuracy of the present theory, cylindrical bending of composite plates under sinusoidal loading is considered. The plate is simply supported at the ends $x_1 = 0$ and l and is infinitely long in the x_2-direction. The prescribed boundary conditions on the top and bottom surfaces of the plate are:

$$T_1^+ = 0, \quad T_3^+ = q \sin\frac{\pi x_1}{l} \qquad \text{on} \qquad x_3 = \frac{h}{2} \tag{22a}$$

$$T_1^- = T_3^- = 0 \qquad \text{on} \qquad x_3 = -\frac{h}{2}. \tag{22b}$$

The boundary conditions for the simply supported ends are, from (20):

$$U_3 = \Psi_3 = S_3 = \xi_3 = 0 \qquad \text{at} \qquad x_1 = 0, l \tag{23a}$$

$$N_{11} = M_{11} = Z_{11} = L_{11} = P_{11} = 0 \qquad \text{at} \qquad x_1 = 0, l. \tag{23b}$$

Using surface boundary conditions (22), the equilibrium eqns (14) for cylindrical bending reduce to:

$$N_{11,1} = 0 \tag{24a}$$

$$N_{13,1} + q\sin\frac{\pi x_1}{l} = 0 \tag{24b}$$

$$M_{11,1} - N_{31} = 0 \tag{24c}$$

$$M_{13,1}-N_{33}+\frac{h}{2}q\sin\frac{\pi x_1}{l}=0 \tag{24d}$$

$$Z_{11,1}-K_{31}=0 \tag{24e}$$

$$Z_{13,1}-K_{33}-q\sin\frac{\pi x_1}{l}=0 \tag{24f}$$

$$L_{11,1}-3M_{31}=0 \tag{24g}$$

$$L_{13,1}-3M_{33}+\frac{h^2}{4}q\sin\frac{\pi x_1}{l}=0 \tag{24h}$$

$$P_{11,1}-5L_{31}-\frac{h^2}{4}N_{31}=0. \tag{24i}$$

From the boundary condition $N_{11}=0$ at $x_1=0,l$, eqn (24a) implies that

$$N_{11}=0. \tag{25}$$

Next, eqns (15a, c) are expressed in terms of the displacement variables $U_1,\ldots,\xi_3$. To this end, the constitutive equations (16) and (17), for the cylindrical bending analysis, can be rewritten in the following vector form:

$$\underset{\sim}{Q}_1-\frac{1}{h^2}\bar{\underset{\sim}{L}}_1+h[A_1]\underset{\sim}{T}_1=\underset{\sim}{\lambda}_1 \tag{26a}$$

$$\frac{1}{h}\underset{\sim}{R}_1+h[B_1]\underset{\sim}{T}_1=\underset{\sim}{\lambda}_2 \tag{26b}$$

$$\underset{\sim}{Q}_1-\frac{7}{4}\frac{1}{h^2}\bar{\underset{\sim}{L}}_1+\frac{5}{2}h[A_1]\underset{\sim}{T}_1=\underset{\sim}{\lambda}_3 \tag{26c}$$

$$[TQ_1]\underset{\sim}{Q}_1+\frac{1}{h}[TR_1]\underset{\sim}{R}_1-\frac{5}{2}\frac{1}{h^2}[TQ_1]\bar{\underset{\sim}{L}}_1=h[C_1]\underset{\sim}{T}_1 \tag{26d}$$

and

$$\underset{\sim}{Q}_3-\frac{1}{h^2}\bar{\underset{\sim}{L}}_3+h[A_1]\underset{\sim}{T}_3=\underset{\sim}{\kappa}_1 \tag{27a}$$

$$\frac{1}{h}\underset{\sim}{R}_3-\frac{1}{h^3}\bar{\underset{\sim}{L}}_3-\frac{2}{7}h[B_1]\underset{\sim}{T}_3=\underset{\sim}{\kappa}_2 \tag{27b}$$

$$\underset{\sim}{Q}_3-\frac{7}{4}\frac{1}{h^2}\bar{\underset{\sim}{L}}_3+\frac{5}{2}h[A_1]\underset{\sim}{T}_3=\underset{\sim}{\kappa}_3 \tag{27c}$$

$$\frac{1}{h}\underset{\sim}{R}_3-\frac{75}{64}\frac{1}{h^3}\bar{\underset{\sim}{L}}_3-\frac{5}{12}h[B_1]\underset{\sim}{T}_3=\underset{\sim}{\kappa}_4 \tag{27d}$$

$$[TQ_3]\underset{\sim}{Q}_3+\frac{1}{h}[TR_3]\underset{\sim}{R}_3-\frac{5}{2}\frac{1}{h^2}[TQ_3]\bar{\underset{\sim}{L}}_3-\frac{35}{24}\frac{1}{h^3}[TR_3]\bar{\underset{\sim}{L}}_3=h[C_3]\underset{\sim}{T}_3 \tag{27e}$$

where

$$\bar{\underset{\sim}{I}}_i \equiv \frac{8\underset{\sim}{I}_i}{n^{(k)2}} \quad i = 1,3 \qquad \text{and} \qquad \bar{\underset{\sim}{I}}_3 \equiv \frac{32}{5n^{(k)2}}\underset{\sim}{I}_3. \tag{28}$$

The matrices $[A_1], \ldots, [C_3]$ and vectors $\underset{\sim}{\lambda}_1, \ldots, \underset{\sim}{\kappa}_4$ are given in the Appendix. The vector equations (26a, b, c) and (27a, b, c, d) have N-components, while the vector equations (26d) and (27e) have $(N-1)$ components. Matrices $[A_1], \ldots, [C_3]$ depend on the volume fractions $n^{(k)}$ and elastic constants $C_{55}^{(k)}$, $C_{13}^{(k)}$ and $C_{33}^{(k)}$, while the vectors $\underset{\sim}{\lambda}_1, \ldots, \underset{\sim}{\kappa}_4$ contain the displacement variables $U_1, \ldots, \xi_3$.

Equations (26) are easily solved by substituting $\underset{\sim}{Q}_1, \frac{1}{h}\underset{\sim}{R}_1$ and $\frac{1}{h^2}\bar{\underset{\sim}{I}}_1$ in terms of $\underset{\sim}{T}_1$ from (26a, b, c) into (26d). This yields a new equation involving $\underset{\sim}{T}_1$ only, which can thus be solved for $\underset{\sim}{T}_1$. Then by back substitution expressions for $\underset{\sim}{Q}_1, \frac{1}{h}\underset{\sim}{R}_1$ and $\frac{1}{h^2}\bar{\underset{\sim}{I}}_1$ in terms of $\underset{\sim}{\lambda}_1, \underset{\sim}{\lambda}_2$ and $\underset{\sim}{\lambda}_3$ are obtained. Proceeding in a similar manner with (27a, b, c, d) $\underset{\sim}{Q}_3, \frac{1}{h}\underset{\sim}{R}_3, \frac{1}{h^2}\bar{\underset{\sim}{I}}_3$ and $\frac{1}{h^3}\bar{\underset{\sim}{I}}_3$ are determined in terms of $\underset{\sim}{\kappa}_1, \underset{\sim}{\kappa}_2, \underset{\sim}{\kappa}_3$ and $\underset{\sim}{\kappa}_4$. These expressions are:

$$\begin{bmatrix} \underset{\sim}{Q}_1 \\ \frac{1}{h^2}\bar{\underset{\sim}{I}}_1 \end{bmatrix} = \begin{bmatrix} \left(\frac{7}{3}[I]-[AQ_1]\right) & -\left(\frac{4}{3}[I]-2[AQ_1]\right) \\ \left(\frac{4}{3}[I]-2[AQ_1]\right) & -\left(\frac{4}{3}[I]-4[AQ_1]\right) \end{bmatrix} \begin{bmatrix} \underset{\sim}{\lambda}_1 \\ \underset{\sim}{\lambda}_3 \end{bmatrix} + \begin{bmatrix} [AR_1] \\ 2[AR_1] \end{bmatrix} \underset{\sim}{\lambda}_2 \tag{29a}$$

$$\frac{1}{h}\underset{\sim}{R}_1 = [BQ_1](\underset{\sim}{\lambda}_1 - 2\underset{\sim}{\lambda}_3) + ([I]-[BR_1])\underset{\sim}{\lambda}_2 \tag{29b}$$

and

$$\begin{bmatrix} \underset{\sim}{Q}_3 \\ \frac{1}{h^2}\bar{\underset{\sim}{I}}_3 \end{bmatrix} = \begin{bmatrix} \left(\frac{7}{3}[I]-[AQ_3]\right) & -\left(\frac{4}{3}[I]-2[AQ_3]\right) \\ \left(\frac{4}{3}[I]-2[AQ_3]\right) & -\left(\frac{4}{3}[I]-4[AQ_3]\right) \end{bmatrix} \begin{bmatrix} \underset{\sim}{\kappa}_1 \\ \underset{\sim}{\kappa}_3 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} [AR_3] \\ 2[AR_3] \end{bmatrix} (8\underset{\sim}{\kappa}_4 - 5\underset{\sim}{\kappa}_2) \tag{30a}$$

$$\begin{bmatrix} \frac{1}{h}\underset{\sim}{R}_3 \\ \frac{1}{h^3}\bar{\underset{\sim}{I}}_3 \end{bmatrix} = \begin{bmatrix} \left(\frac{75}{11}[I]+\frac{50}{63}[BR_3]\right) & -\left(\frac{64}{11}[I]+\frac{80}{63}[BR_3]\right) \\ \left(\frac{64}{11}[I]+\frac{80}{63}[BR_3]\right) & -\left(\frac{64}{11}[I]+\frac{128}{63}[BR_3]\right) \end{bmatrix} \begin{bmatrix} \underset{\sim}{\kappa}_2 \\ \underset{\sim}{\kappa}_4 \end{bmatrix} - \frac{1}{21}\begin{bmatrix} 10[BQ_3] \\ 16[BQ_3] \end{bmatrix} (2\underset{\sim}{\kappa}_3 - \underset{\sim}{\kappa}_1) \tag{30b}$$

where $[I]$ is the $N \times N$ identity matrix and

$$\begin{bmatrix} [AQ_i] & [AR_i] \\ [BQ_i] & [BR_i] \end{bmatrix} = \begin{bmatrix} [A_1] \\ [B_1] \end{bmatrix} [TV_i]\,[[TQ_i] \quad [TR_i]] \quad i = 1,3 \tag{31a}$$

with

$$[TV_1] = (4[TQ_1][A_1] + [TR_1][B_1] + [C_1])^{-1} \tag{31b}$$

$$[TV_3] = (4[TQ_3][A_1] - \frac{40}{63}[TR_3][B_1] + [C_3])^{-1}. \tag{31c}$$

$[AQ_i], \ldots, [BR_i]$ are $N \times N$ matrices, while $[TV_i]$ are $(N-1) \times (N-1)$ matrices. By inserting (29) and (30) into (15c) and (21) the appropriate constitutive relations for the cylindrical bending problem in terms of the displacement variables $U_1, \ldots, \xi_3$ and their derivatives with respect to x_1 are obtained.

The form of the dependence on the displacement variables $U_1, \ldots, \xi_3$ of the constitutive equations thus obtained and the nature of the applied load suggest the following expressions for the displacements:

$$\begin{bmatrix} U_1 \\ \Psi_1 \\ S_1 \\ \xi_1 \\ \phi_1 \end{bmatrix} = \begin{bmatrix} h\hat{U}_1 \\ \hat{\Psi}_1 \\ h\hat{S}_1 \\ \hat{\xi}_1/h \\ \hat{\phi}_1/h^2 \end{bmatrix} \cos \pi \frac{x_1}{l} \quad \text{and} \quad \begin{bmatrix} U_3 \\ \Psi_3 \\ S_3 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} h\hat{U}_3 \\ \hat{\Psi}_3 \\ h\hat{S}_3 \\ \hat{\xi}_3/h \end{bmatrix} \sin \pi \frac{x_1}{l} \tag{32}$$

where the "$\hat{}$" quantities are nondimensional by definition. It is easily proven that the boundary conditions (23) are satisfied when (32) are substituted therein.

Finally, inserting (32) into the constitutive equations obtained in the manner described above and these in turn into the equilibrium equations (24) and (25) yields a system of nine algebraic equations with the nine nondimensional quantities $\hat{U}_1, \ldots, \hat{\xi}_3$ as unknowns. This system is conveniently written in matrix form as

$$[B]\underset{\sim}{U} = \underset{\sim}{F} \tag{33}$$

where

$$\underset{\sim}{U} = [\hat{U}_1 \hat{\Psi}_1 \hat{S}_1 \hat{\xi}_1 \hat{\phi}_1 \hat{U}_3 \hat{\Psi}_3 \hat{S}_3 \hat{\xi}_3]^T \tag{34a}$$

$$\underset{\sim}{F} = \left[0, q, 0, \frac{1}{2}q, 0, -q, 0, \frac{1}{4}q, 0\right]^T \tag{34b}$$

and $[B]$ is a 9×9 matrix.

6. NUMERICAL RESULTS

In order to assess the accuracy of the present theory the problem of the cylindrical bending of an infinitely long strip under sinusoidal loading is examined. The exact elasticity solution has been given by Pagano[1], where a three layer cross-ply laminate was considered, the 0° layers being at the outer surfaces of the laminate. The elastic properties are:

$$\text{for the 0° layers} \quad \frac{\tilde{C}_{11}}{E_T} = 25.062657, \quad \frac{C_{13}}{E_T} = 0.335570$$
$$\frac{C_{33}}{E_T} = 1.071141, \quad \frac{C_{55}}{E_T} = 0.5; \tag{35a}$$

$$\text{and for the 90° layers} \quad \frac{\tilde{C}_{11}}{E_T} = 1.002506, \quad \frac{C_{13}}{E_T} = 0.271141$$
$$\frac{C_{33}}{E_T} = 1.071141, \quad \frac{C_{55}}{E_T} = 0.2 \tag{35b}$$

where E_T is a reference modulus.

We follow Pagano's[1] nondimensionalization and write the displacements and stresses in the form

$$\bar{u}_1^{(k)} = \left(\frac{E_T}{q}\right)\frac{u_1^{(k)}(0, x_3)}{h}, \qquad \bar{u}_3^{(k)} = \left(\frac{E_T}{q}\right)\frac{100h^3}{l^4}u_3^{(k)}\left(\frac{l}{2}, 0\right)$$
$$\bar{\sigma}_{11}^{(k)} = \frac{1}{q}\sigma_{11}^{(k)}\left(\frac{l}{2}, x_3\right). \tag{36}$$

Also

$$\bar{x}_3 = \frac{x_3}{h}, \qquad S = \frac{l}{h}. \tag{37}$$

In the various curves the solid line represents the exact solution while the results of the present theory are shown by a broken line. Also shown, for comparison purposes, are the results given by the first-order zig-zag model[12] and Lo, Christensen and Wu's high-order theory (LCW)[7], which are represented by a dashed-dotted line and dotted solid line, respectively. Symmetric 3-, 5- and 9-ply laminates and asymmetric 4- and 8-ply laminates were examined, to test the present theory.

For a symmetric 3-ply laminate (0/90/0) with layers of equal thickness, Table 1 shows the values of the central deflection $\bar{u}_3$ obtained from the different theories for a span-to-thickness ratio S of 4 and 6. As observed the present high-order theory correctly predicts the central deflection $\bar{u}_3$ to the first two decimal digits, while the first-order zig-zag model gives a better result than LCW. The variation of the in-plane displacement $\bar{u}_1$ across the plate thickness is compared in Fig. 2a for $S = 4$, where it is seen that the curves for the present theory and the exact solution are almost identical. This improvement is also reflected in the variation of the in-plane stress $\bar{\sigma}_{11}$ across the plate thickness, as shown in Fig. 2b. Very close agreement is found between Pagano's exact solution and the present theory, which has improved upon Lo, Christensen and Wu's high-order theory, especially at and in the neighborhood of the interfaces.

The present theory was next tested for a symmetric 5-ply laminate (0/90/0/90/0) with layers of equal thickness. The central deflection $\bar{u}_3$ for a span-to-thickness ratio S of 4 and 6, is shown in Table 1 where close agreement with the exact solution is observed. The variations across the plate thickness of in-plane variables $\bar{u}_1^{(k)}$ and $\bar{\sigma}_{11}^{(k)}$ are compared in Figs 3 and 4. The curves for the present high-order theory and the exact solution are again almost identical. In particular, it is seen that the present theory has considerably improved upon Lo, Christensen and Wu's model in the interior layers of the plate.

To further assess the accuracy of the present high-order theory the more difficult case of a symmetric 9-layer cross-ply laminate (0/90/0/90/0/90/0/90/0) was considered. The 0° layers have equal thickness $h/10$ while the 90° layers have equal thickness $h/8$. The results for the central deflection $\bar{u}_3$ are given in Table 1 for $S = 4$ and 6 where again close agreement with the exact solution is observed. The variations across the plate thickness of the in-plane displacement $\bar{u}_1$ and normal stress $\bar{\sigma}_{11}$ are shown in Figs 5 and 6, for $S = 4$ and 6,

Table 1. Central deflection $\bar{u}_3$ for symmetric cross-ply laminates in cylindrical bending under sinusoidal loading

	$S = 4$			$S = 6$		
Number of layers N	3	5	9	3	5	9
Exact solution[1]	2.887	3.044	3.324	1.635	1.721	1.929
Present theory	2.881	3.032	3.313	1.634	1.716	1.921
First-order zig-zag[12]	2.907	3.018	3.231	1.636	1.702	1.875
LCW[7]	2.687	2.597	2.835	1.514	1.507	1.708

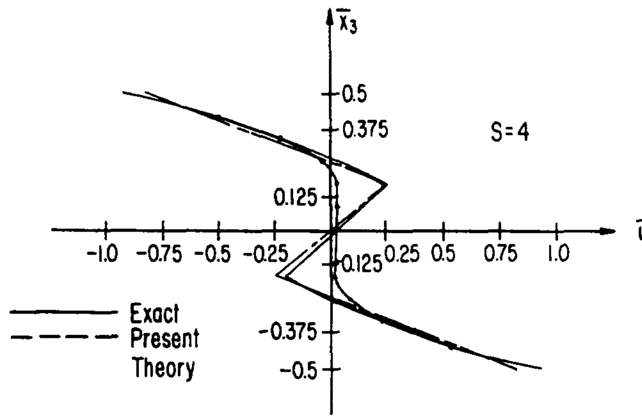

Fig. 2a. Thickness variation of in-plane displacement $\bar{u}_1^{(k)}$ of a symmetric 3-layer cross-ply laminate for $S = 4$.

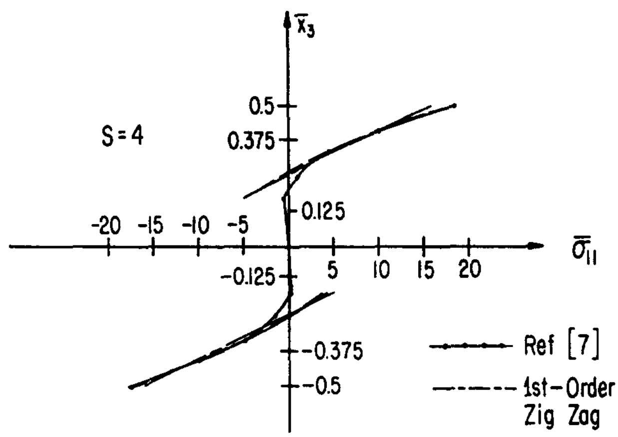

Fig. 2b. Thickness variation of normal stress $\bar{\sigma}_{11}^{(k)}$ of a symmetric 3-layer cross-ply laminate for $S = 4$.

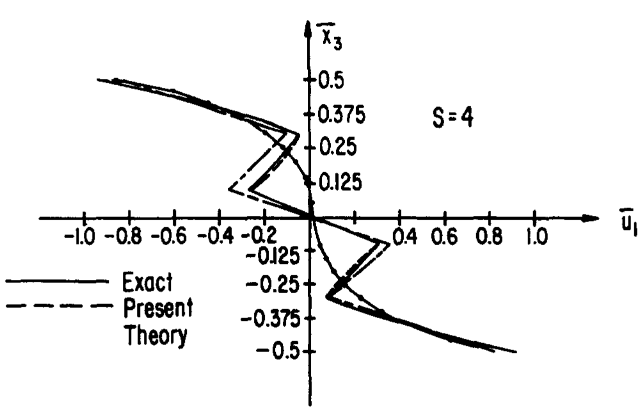

Fig. 3a. Thickness variation of in-plane displacement $\bar{u}_1^{(k)}$ of a symmetric 5-layer cross-ply laminate for $S = 4$.

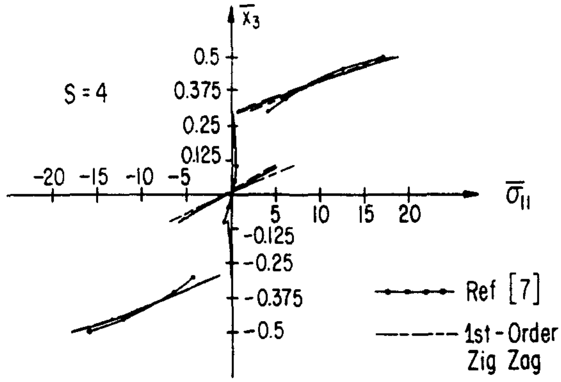

Fig. 3b. Thickness variation of normal stress $\bar{\sigma}_{11}^{(k)}$ of a symmetric 5-layer cross-ply laminate for $S = 4$.

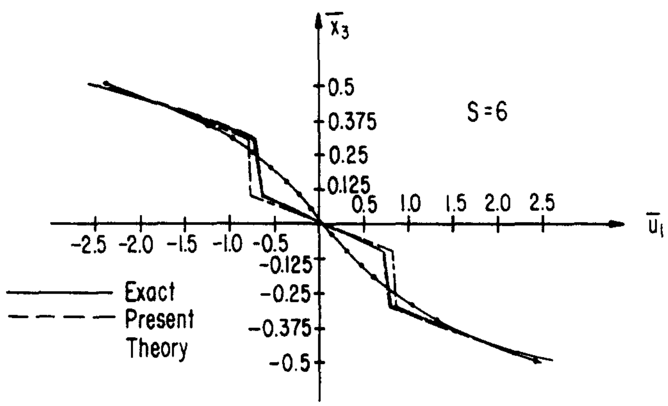

Fig. 4a. Thickness variation of in-plane displacement $\bar{u}_1^{(k)}$ of a symmetric 5-layer cross-ply laminate for $S = 6$.

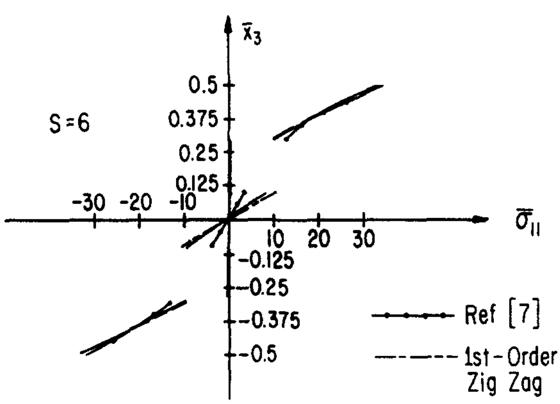

Fig. 4b. Thickness variation of normal stress $\bar{\sigma}_{11}^{(k)}$ of a symmetric 5-layer cross-ply laminate for $S = 6$.

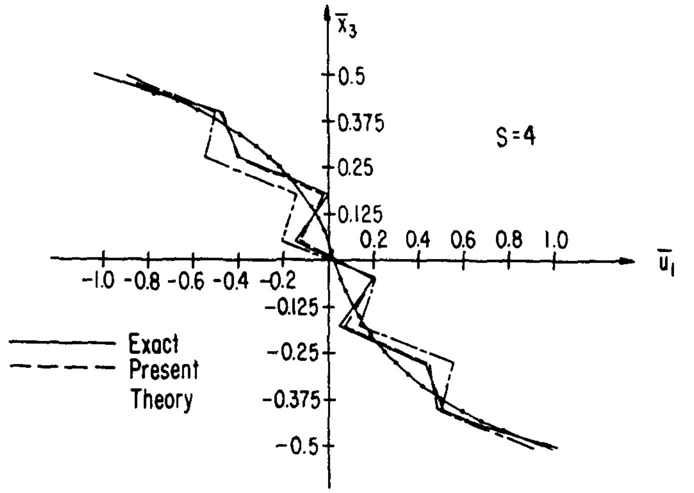

Fig. 5a. Thickness variation of in-plane displacement $\bar{u}_1^{(k)}$ of a symmetric 9-layer cross-ply laminate for $S = 4$.

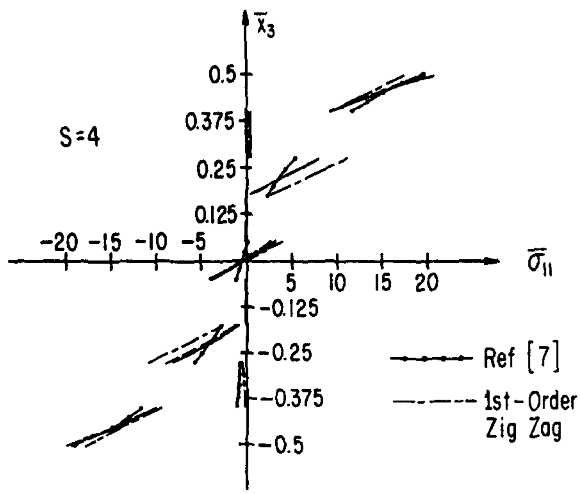

Fig. 5b. Thickness variation of normal stress $\bar{\sigma}_{11}^{(k)}$ of a symmetric 9-layer cross-ply laminate for $S = 4$.

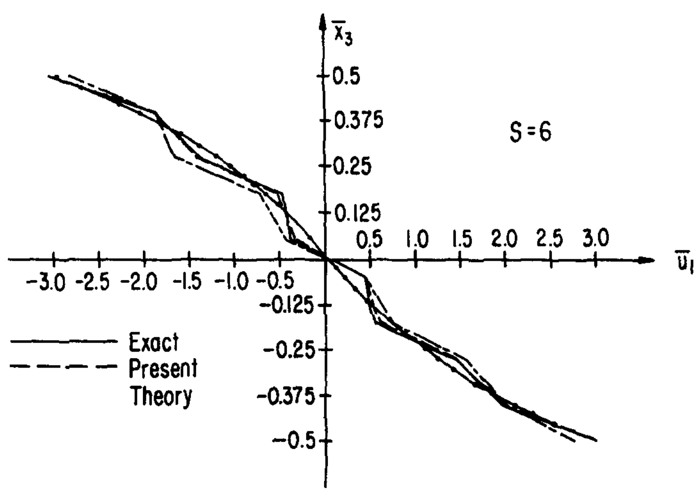

Fig. 6a. Thickness variation of in-plane displacement $\bar{u}_1^{(k)}$ of a symmetric 9-layer cross-ply laminate for $S = 6$.

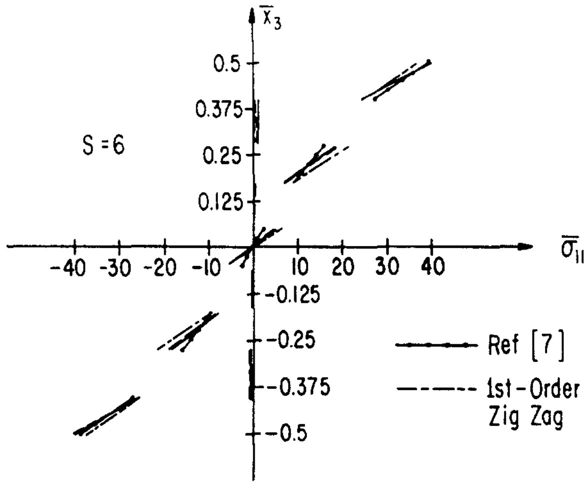

Fig. 6b. Thickness variation of normal stress $\bar{\sigma}_{11}^{(k)}$ of a symmetric 9-layer cross-ply laminate for $S = 6$.

respectively. There the discrepancies between the first-order zig-zag theory and the exact solution are more pronounced than in the 3- and 5-layer cases, as expected. However, the results of the present theory are still very good when compared to the exact solution.

Finally, asymmetric 4 and 8 cross-ply laminates, with layers of equal thickness, were examined. The present theory predicts accurately the central deflection $\bar{u}_3$. These results are given in Table 2 for a span-to-thickness ratio S of 4 and 6. The variation across the plate thickness of the in-plane displacement $\bar{u}_1^{(k)}$ and normal stress $\bar{\sigma}_{11}^{(k)}$ are shown in Figs 7, 8 and 9 for $S = 4$ and 6. From the curves for $\bar{u}_1^{(k)}$, it is seen that the first-order zig-zag theory deviates significantly from the exact solution at the bottom layer of the plate. On the other hand, the discrepancies between LCW and the exact solution, for both $\bar{u}_1^{(k)}$ and $\bar{\sigma}_{11}^{(k)}$ are more pronounced in the interior layers of the plate, while the present high-order theory is still in very good agreement with the exact solution.

7. CONCLUSION

A high-order laminated plate theory, which accurately predicts in-plane responses of symmetric and asymmetric laminates, was developed with the help of Reissner's new mixed variational principle[10]. The improvement was achieved by including a zig-zag shaped C^0 function in the in-plane displacement variations across the plate thickness, as proposed by Murakami[12], while the high-order variation is accounted for by using Legendre polynomials. The accuracy of the theory was examined for the case of cylindrical bending of an infinitely long strip and compared with the exact elasticity solution given by Pagano[1]. The results for the central deflection and in-plane displacements and normal stresses for several symmetric and asymmetric cross-ply laminates indicate that the theory very accurately predicts these in-plane responses even for small span-to-thickness ratios. In all the cases

Table 2. Central deflection $\bar{u}_3$ for asymmetric cross-ply laminates in cylindrical bending under sinusoidal loading

	$S = 4$		$S = 6$	
Number of layers N	4	8	4	8
Exact solution[1]	4.181	3.724	2.562	2.224
Present theory	4.105	3.625	2.519	2.181
First-order zig-zag[12]	3.316	3.225	2.107	1.934
LCW[7]	3.587	3.189	2.242	1.979

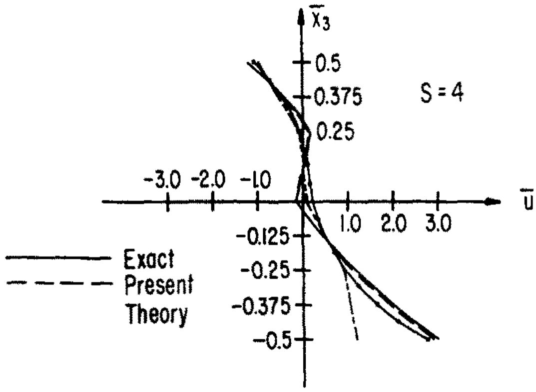

Fig. 7a. Thickness variation of in-plane displacement $\bar{u}_1^{(k)}$ of an asymmetric 4-layer cross-ply laminate for $S = 4$.

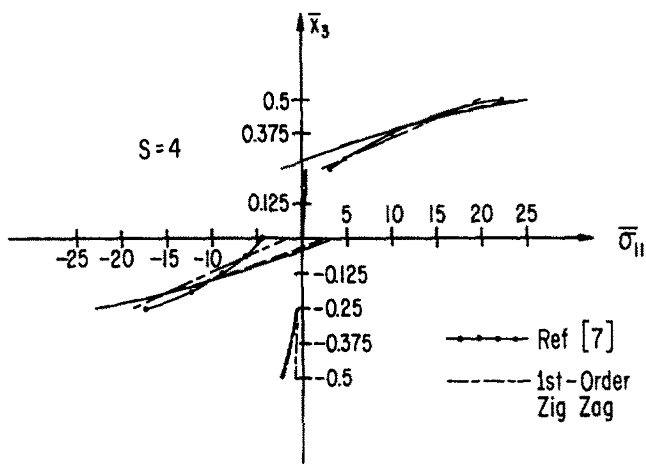

Fig. 7b. Thickness variation of normal stress $\bar{\sigma}_{11}^{(k)}$ of an asymmetric 4-layer cross-ply laminate for $S = 4$.

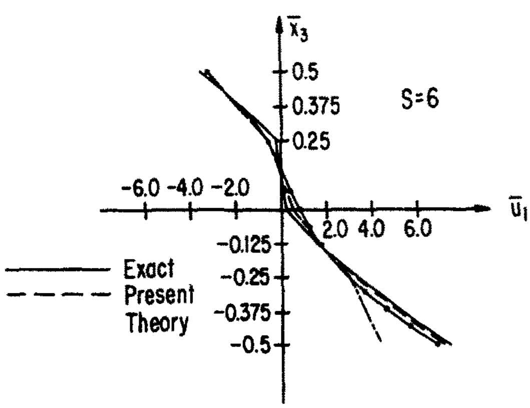

Fig. 8a. Thickness variation of in-plane displacement $\bar{u}_1^{(k)}$ of an asymmetric 4-layer cross-ply laminate for $S = 6$.

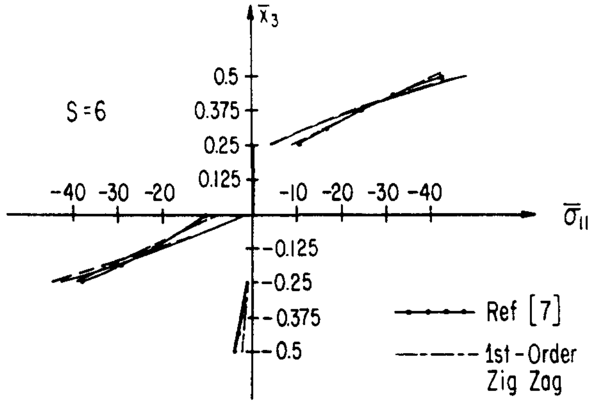

Fig. 8b. Thickness variation of normal stress $\bar{\sigma}_{11}^{(k)}$ of an asymmetric 4-layer cross-ply laminate for $S = 6$.

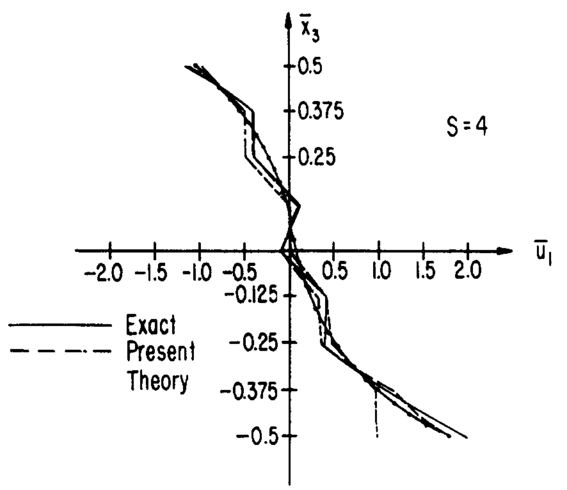

Fig. 9a. Thickness variation of in-plane displacement $\bar{u}_1^{(k)}$ of an asymmetric 8-layer cross-ply laminate for $S = 4$.

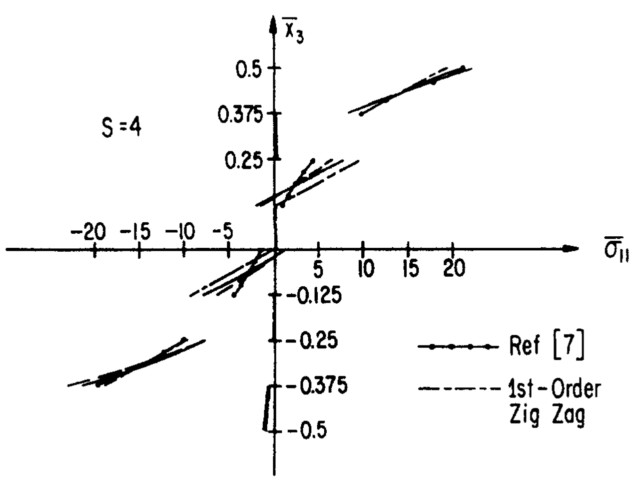

Fig. 9b. Thickness variation of normal stress $\bar{\sigma}_{11}^{(k)}$ of an asymmetric 8-layer cross-ply laminate for $S = 4$.

considered, the proposed theory gave better in-plane responses than the Lo, Christensen and Wu high-order theory, especially in the interior layers of the plate. It was also observed that for symmetric laminates, the first-order zig-zag model[12] predicts more accurately the central deflection than the Lo, Christensen and Wu high-order theory.

Acknowledgement—The research was supported by the Office of Naval Research under Contract N00014-84-K-0468 to the University of California at San Diego. The authors are grateful to Dr A. Kushner for his support. Also, the author is thankful to Professors E. Reissner and G. A. Hegemier for their advice and comments on the manuscript.

REFERENCES

1. N. J. Pagano, Exact solutions for composite laminates in cylindrical bending. *J. Comp. Mat.* **3**, 398 (1969).
2. N. J. Pagano, Exact solutions for rectangular bidirectional composites and sandwich plates. *J. Comp. Mat.* **4**, 20 (1970).
3. N. J. Pagano and S. J. Hatfield, Elastic behavior of multilayered bidirectional composites. *AIAA J.* **10**, 931 (1972).
4. P. C. Yang, C. H. Norris and Y. Stavsky, Elastic wave propagation in heterogeneous plates. *Int. J. Solids Structures* **2**, 665 (1966).
5. J. M. Whitney and N. J. Pagano, Shear deformation in heterogeneous anisotropic plates. *ASME J. Appl. Mech.* **37**, 1031(1970).
6. J. M. Whitney, Stress analysis of thick laminated composite and sandwich plates. *J. Comp. Mat.* **6**, 426 (1972).
7. K. H. Lo, R. M. Christensen and E. M. Wu, A high-order theory of plate deformation—part 2: laminated plates. *ASME J. Appl. Mech.* **44**, 669 (1977).
8. J. N. Reddy, A simple higher-order theory for laminated composite plates. *ASME J. Appl. Mech.* **51**, 745 (1984).
9. F. B. Hildebrand, E. Reissner and G. B. Thomas, Notes on the foundations of the theory of small displacements of orthotropic shells. *NACA Technical Note* No. 1833 (1949).
10. E. Reissner, On a certain mixed variational theorem and a proposed application. *Int. J. Num. Meth. Eng.* **20**, 1366 (1984).
11. E. Reissner, On a mixed variational theorem and on shear deformable plate theory. *Int. J. Num. Meth. Eng.* **23**, 193 (1986).
12. H. Murakami, A laminated composite plate theory with improved in-plane responses. *Proceedings of the 1985 PVP Conference, ASME, PVP*, Vol. 98-2, pp. 257–263 (1985), also *ASME J. Appl. Mech.* **53**, 661 (1986).

APPENDIX

Matrices $[N_u], \ldots, [P_\phi]$ *in eqn* (21)

$$[N_u] = \begin{bmatrix} D_2 & D_2' & 0 \\ D_2' & D_2'' & 0 \\ 0 & 0 & D_2''' \end{bmatrix}, \quad [N_\Psi] = \begin{bmatrix} C_1 & C_1' & 0 \\ C_1' & C_1'' & 0 \\ 0 & 0 & C_1''' \end{bmatrix},$$

$$[N_\zeta] = \begin{bmatrix} C_8 & C_8' & 0 \\ C_8' & C_8'' & 0 \\ 0 & 0 & C_8''' \end{bmatrix}, \quad [N_\phi] = \begin{bmatrix} C_9 & C_9' & 0 \\ C_9' & C_9'' & 0 \\ 0 & 0 & C_9''' \end{bmatrix}$$

$$[M_\Psi] = \begin{bmatrix} C_2 & C_2' & 0 \\ C_2' & C_2'' & 0 \\ 0 & 0 & C_2''' \end{bmatrix}, \quad [M_s] = \begin{bmatrix} C_3 & C_3' & 0 \\ C_3' & C_3'' & 0 \\ 0 & 0 & C_3''' \end{bmatrix},$$

$$[M_\zeta] = \begin{bmatrix} C_7 & C_7' & 0 \\ C_7' & C_7'' & 0 \\ 0 & 0 & C_7''' \end{bmatrix}, \quad [M_\phi] = \begin{bmatrix} C_6 & C_6' & 0 \\ C_6' & C_6'' & 0 \\ 0 & 0 & C_6''' \end{bmatrix} \quad \text{(A1)}$$

$$[Z_\zeta] = \begin{bmatrix} D_3 & D_3' & 0 \\ D_3' & D_3'' & 0 \\ 0 & 0 & D_3''' \end{bmatrix}, \quad [Z_\phi] = \begin{bmatrix} D_1 & D_1' & 0 \\ D_1' & D_1'' & 0 \\ 0 & 0 & D_1''' \end{bmatrix},$$

$$[L_\zeta] = \begin{bmatrix} D_5 & D_5' & 0 \\ D_5' & D_5'' & 0 \\ 0 & 0 & D_5''' \end{bmatrix}, \quad [L_\phi] = \begin{bmatrix} F_3 & F_3' & 0 \\ F_3' & F_3'' & 0 \\ 0 & 0 & F_3''' \end{bmatrix}$$

$$[P_\phi] = \begin{bmatrix} F_4 & F_4' & 0 \\ F_4' & F_4'' & 0 \\ 0 & 0 & F_4''' \end{bmatrix}$$

where

$$\begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = \sum_k \tilde{C}_{11}^{(k)} \begin{bmatrix} n_o^{(k)} n^{(k)} \\ n^{(k)3}/12 + n_o^{(k)2} n^{(k)} \\ (-1)^k n^{(k)2}/6 \end{bmatrix}, \quad \begin{bmatrix} D_2 \\ D_3 \end{bmatrix} = \sum_k \tilde{C}_{11}^{(k)} \begin{bmatrix} n^{(k)} \\ (-1)^k n_o^{(k)} n^{(k)2}/2 \end{bmatrix} \tag{A2}$$

and

$$C_6 = \frac{5}{2}C_5 - \frac{3}{8}C_2, \quad C_7 = \frac{3}{2}C_4 - \frac{1}{8}C_1, \quad C_8 = \frac{3}{2}C_2 - \frac{1}{8}D_2, \quad C_9 = \frac{5}{2}C_4 - \frac{3}{8}C_1$$

$$D_1 = D_4 - \frac{3}{8}C_3, \quad D_5 = \frac{9}{4}C_5 - \frac{3}{8}C_2 + \frac{1}{64}D_2$$

$$F_3 = \frac{15}{4}F_1 - \frac{7}{8}C_4 + \frac{3}{64}C_1, \quad F_4 = \frac{25}{4}F_2 - \frac{15}{8}C_5 + \frac{9}{64}C_2 \tag{A3}$$

where

$$\begin{bmatrix} C_4 \\ C_5 \end{bmatrix} = \sum_k \tilde{C}_{11}^{(k)} \begin{bmatrix} \frac{1}{4} n_o^{(k)} n^{(k)3} + n_o^{(k)3} n^{(k)} \\ \frac{1}{80} n^{(k)5} + \frac{1}{2} n_o^{(k)2} n^{(k)3} + n_o^{(k)4} n^{(k)} \end{bmatrix}, \quad D_4 = \sum_k \tilde{C}_{11}^{(k)} (-1)^k \left(\frac{n^{(k)4}}{16} + \frac{5}{4} n_o^{(k)2} n^{(k)2} \right)$$

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \sum_k \tilde{C}_{11}^{(k)} \begin{bmatrix} \frac{n_o^{(k)} n^{(k)5}}{16} + \frac{5}{6} n_o^{(k)3} n^{(k)3} + n_o^{(k)5} n^{(k)} \\ \frac{n^{(k)7}}{448} + \frac{3}{16} n_o^{(k)2} n^{(k)5} + \frac{5}{4} n_o^{(k)4} n^{(k)3} + n_o^{(k)6} n^{(k)} \end{bmatrix}. \tag{A4}$$

The ( )′, ( )″ and ( )‴ quantities can be obtained from (A1, 2, 3, 4) by replacing therein $\tilde{C}_{11}^{(k)}$ by $\tilde{C}_{12}^{(k)}$, $\tilde{C}_{22}^{(k)}$ and $\tilde{C}_{66}^{(k)}$, respectively, where k ranges from 1 to N.

Matrix $[C]^{(k)}$ *and vectors* $\underset{\sim}{V}^N, \ldots, \underset{\sim}{V}^P$ *in eqn* (21)

$$\underset{(15\times 5)}{[C]^{(k)}} = \begin{bmatrix} \underset{\sim}{c} & & & & \\ & \underset{\sim}{c} & & 0 & \\ & & \underset{\sim}{c} & & \\ & 0 & & \underset{\sim}{c} & \\ & & & & \underset{\sim}{c} \end{bmatrix}^{(k)} \quad \text{where } \underset{\sim}{c}^{(k)} = [C_{13}/C_{33} \quad C_{23}/C_{33} \quad 0]^{(k)T} \tag{A5}$$

$$\underset{\sim}{V}^N = [1,0,0,0]; \quad \underset{\sim}{V}^M = [n_o^{(k)},1,0,0]; \quad \underset{\sim}{V}^Z = \left[0,(-1)^k \frac{2}{n^{(k)}},0,0\right]$$

$$\underset{\sim}{V}^L = \left[\frac{1}{2}\left(3n_o^{(k)2} - \frac{1}{4}\right), 3n_o^{(k)}, \frac{3}{2}, 0\right]; \quad \underset{\sim}{V}^P = \left[\frac{1}{2}\left(5n_o^{(k)3} - \frac{3n_o^{(k)}}{4}\right), \frac{3}{2}\left(5n_o^{(k)2} - \frac{1}{4}\right), \frac{15}{2}n_o^{(k)}, \frac{5}{2}\right]. \tag{A6}$$

Matrices $[A_1], \ldots, [C_3]$ *in eqns* (26) *and* (27)

$$\underset{N\times(N-1)}{[A_1]} = \frac{1}{30}\begin{bmatrix} & & 0 \\ n^{(k)} & n^{(k)} & \\ 0 & & \end{bmatrix}, \quad \underset{(N\times N-1)}{[B_1]} = \frac{1}{40}\begin{bmatrix} & & 0 \\ -n^{(k)2} & n^{(k)2} & \\ 0 & & \end{bmatrix},$$

$$\underset{(N-1)\times(N-1)}{[C_1]} = \frac{1}{126}\begin{bmatrix} & & 0 \\ \frac{-n^{(k)}}{C_{55}^{(k)}} & 8\left(\frac{n^{(k)}}{C_{55}^{(k)}} + \frac{n^{(k+1)}}{C_{55}^{(k+1)}}\right) & \frac{-n^{(k+1)}}{C_{55}^{(k+1)}} \\ 0 & & \end{bmatrix}$$

$$\underset{(N-1)\times N}{[TQ_1]} = \frac{1}{12}\begin{bmatrix} & 0 \\ \frac{-1}{C_{55}^{(k)}} & \frac{-1}{C_{55}^{(k+1)}} \\ 0 & \end{bmatrix}, \quad \underset{(N-1)\times N}{[TR_1]} = \frac{3}{7}\begin{bmatrix} & 0 \\ \frac{-1}{n^{(k)}C_{55}^{(k)}} & \frac{1}{n^{(k+1)}C_{55}^{(k+1)}} \\ 0 & \end{bmatrix}, \tag{A7}$$

$$\underset{(N-1)\times N}{[TQ_3]} = \frac{11}{12}\begin{bmatrix} & 0 \\ \dfrac{1}{C_{33}^{(k)}} & \dfrac{1}{C_{33}^{(k+1)}} \\ 0 & \end{bmatrix},$$

$$\underset{(N-1)\times N}{[TR_3]} = \frac{-15}{2}\begin{bmatrix} & 0 \\ \dfrac{1}{n^{(k)}C_{33}^{(k)}} & \dfrac{-1}{n^{(k+1)}C_{33}^{(k+1)}} \\ 0 & \end{bmatrix},$$

$$\underset{(N-1)\times(N-1)}{[C_3]} = -\frac{1}{18}\begin{bmatrix} & & 0 \\ \dfrac{n^{(k)}}{C_{33}^{(k)}} & 10\left(\dfrac{n^{(k)}}{C_{33}^{(k)}} + \dfrac{n^{(k+1)}}{C_{33}^{(k+1)}}\right) & \dfrac{n^{(k+1)}}{C_{33}^{(k+1)}} \\ 0 & & \end{bmatrix}.$$

Vectors $\underset{\sim}{\lambda}_1, \ldots, \underset{\sim}{\kappa}_4$ *in eqns* (26) *and* (27)

$$\begin{aligned}
\underset{\sim}{\lambda}_1 &= h(U_{3,1}+\Psi_1)\underset{\sim}{a}_1 + S_1\underset{\sim}{b}_1 + h^2(\Psi_{3,1}+3\xi_1)\underset{\sim}{c}_1 + h^3\xi_{3,1}\underset{\sim}{d}_1 + h^3\phi_1\underset{\sim}{e}_1 \\
\underset{\sim}{\lambda}_2 &= h^2(\Psi_{3,1}+3\xi_1)\underset{\sim}{f}_1 + hS_{3,1}\underset{\sim}{g}_1 + h^3(\xi_{3,1}+5\phi_1)\underset{\sim}{p}_1 \\
\underset{\sim}{\lambda}_3 &= \frac{-9}{7}h^3(\xi_{3,1}+5\phi_1)\underset{\sim}{f}_1 \\
\underset{\sim}{\kappa}_1 &= hU_{1,1}\underset{\sim}{a}_2 + h\Psi_3\underset{\sim}{a}_3 + S_3\underset{\sim}{b}_3 + h^2\xi_3\underset{\sim}{c}_2 + h^2\Psi_{1,1}\underset{\sim}{c}_3 + h^3\xi_{1,1}\underset{\sim}{d}_3 + h^4\phi_{1,1}\underset{\sim}{e}_3 \\
\underset{\sim}{\kappa}_2 &= h^2\xi_3\underset{\sim}{f}_2 + h^2\Psi_{1,1}\underset{\sim}{f}_3 + hS_{1,1}\underset{\sim}{g}_3 + h^3\xi_{1,1}\underset{\sim}{p}_2 + h^4\phi_{1,1}\underset{\sim}{p}_3 \\
\underset{\sim}{\kappa}_3 &= -\frac{315}{44}h^3\xi_{1,1}\underset{\sim}{f}_3 - \frac{525}{44}h^4\phi_{1,1}\underset{\sim}{p}_2 \\
\underset{\sim}{\kappa}_4 &= -h^4\phi_{1,1}\underset{\sim}{s}_3.
\end{aligned} \tag{A8}$$

The kth component of the vectors $\underset{\sim}{a}_1, \ldots, \underset{\sim}{s}_3$ appearing in (A8) are given by

$$\begin{gathered}
a_1^{(k)} = \frac{2}{5}n^{(k)}C_{55}^{(k)}, \quad b_1^{(k)} = \frac{4}{5}(-1)^k C_{55}^{(k)}, \quad c_1^{(k)} = \frac{2}{5}n_o^{(k)}n^{(k)}C_{55}^{(k)}, \quad d_1^{(k)} = \frac{1}{5}\left(3n_o^{(k)2} - \frac{1}{4}\right)n^{(k)}C_{55}^{(k)} \\
e_1^{(k)} = \frac{3}{5}\left(5n_o^{(k)2} - \frac{1}{4}\right)n^{(k)}C_{55}^{(k)}, \quad f_1^{(k)} = \frac{7}{120}n^{(k)3}C_{55}^{(k)}, \quad g_1^{(k)} = \frac{7}{60}(-1)^k n^{(k)2}C_{55}^{(k)}, \quad p_1^{(k)} = \frac{7}{40}n_o^{(k)}n^{(k)3}C_{55}^{(k)} \\
a_2^{(k)} = \frac{2}{5}n^{(k)}C_{13}^{(k)}, \quad a_3^{(k)} = \frac{2}{5}n^{(k)}C_{33}^{(k)}, \quad b_3^{(k)} = \frac{4}{5}(-1)^k C_{33}^{(k)}, \quad c_2^{(k)} = \frac{6}{5}n_o^{(k)}n^{(k)}C_{33}^{(k)}, \quad c_3^{(k)} = \frac{2}{5}n_o^{(k)}n^{(k)}C_{13}^{(k)} \\
d_3^{(k)} = \frac{1}{5}\left(3n_o^{(k)2} - \frac{1}{4}\right)n^{(k)}C_{13}^{(k)}, \quad e_3^{(k)} = \frac{1}{5}\left(5n_o^{(k)3} - \frac{3}{4}n_o^{(k)}\right)n^{(k)}C_{13}^{(k)}, \quad f_2^{(k)} = \frac{11}{350}n^{(k)3}C_{33}^{(k)} \\
f_3^{(k)} = \frac{11}{1050}n^{(k)3}C_{13}^{(k)}, \quad g_3^{(k)} = \frac{11}{525}(-1)^k n^{(k)2}C_{13}^{(k)}, \quad p_2^{(k)} = \frac{11}{350}n_o^{(k)}n^{(k)3}C_{13}^{(k)} \\
p_3^{(k)} = \frac{11}{700}\left(5n_o^{(k)2} - \frac{1}{4}\right)n^{(k)3}C_{13}^{(k)}, \quad s_3^{(k)} = \frac{11}{2688}n^{(k)5}C_{13}^{(k)}.
\end{gathered} \tag{A9}$$